

# NEW CRITERIA OF GENERIC HYPERBOLICITY BASED ON PERIODIC POINTS

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**ABSTRACT.** We prove that, if a mild condition on the hyperbolicity of the periodic points holds for any diffeomorphism in a residual subset of a  $C^1$ -open set  $\mathcal{U}$ , then such set  $\mathcal{U}$  exhibits a residual subset  $\mathcal{A}$  of Axiom A diffeomorphisms. We also prove an analogous result for nonsingular endomorphisms: if a mild expanding condition holds for the periodic set of local diffeomorphisms belonging in a residual subset of a  $C^1$  open set  $\mathcal{U}$ , then  $\mathcal{U}$  exhibits an open and dense subset of expanding maps. For this last result we use non-invertible versions of Ergodic Closing Lemma that we have proved in [5].

## 1. INTRODUCTION

The study of asymptotic rates of growth and conditions over the set of periodic points of a diffeomorphism  $f$  on a compact manifold  $M$ , and its relations with uniform hyperbolic behavior appeared recently in [6]. In that article, the authors define a mild condition of hyperbolicity called NUH (non-uniformly hyperbolic set, inspired in [12], [3], [4]), and proved that if a diffeomorphism is conjugated to a hyperbolic one, and has a NUH set of periodic points, then such diffeomorphism is, itself, hyperbolic. The authors also state analogous results in which the conjugation property is replaced with a shadowing property by periodic points as hypothesis. The main idea in the proof of [6] is that the asymptotic hyperbolicity of a NUH periodic set spreads out to any recurrent point, since (due to the shadowing condition) any recurrent point is shadowed by periodic ones. Moreover, if the NUH periodic set exhibits a dominated splitting then any  $f$ -invariant probability measure has only nonzero Lyapunov exponents. Thus, by [1], this is equivalent to the uniform hyperbolicity of the periodic set closure.

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However, even though dominated splitting is a generic (residual) feature of robustly transitive maximal invariant sets (see [2]), we note that far away from hyperbolic systems the classical, strong notion of shadowing property seems to be somewhat rare (see [7]). Despite this fact, article [6] opened the way for others (for instance, [8]) which use the same idea: the existence of a mild hyperbolicity on the periodic set, plus a shadowing property by periodic points, implies uniform hyperbolicity.

In the present work, we consider systems which could be far from the hyperbolic ones, as we do not assume a priori any hypothesis of shadowing. If we suppose that an open set  $\mathcal{U}$  of such systems exhibits a residual subset of diffeomorphisms whose periodic sets are NUH, then we prove that there also exists a residual subset of  $\mathcal{U}$  whose elements are hyperbolic diffeomorphisms (Axiom A). An analogous criteria (of expansion) based on periodic sets is obtained in the case of nonsingular endomorphisms. In this last case, the periodic set is assumed to be NUE - non uniformly expanding set. Precise definitions and statements of our results are given in the text below, and the more technical situation of NUH endomorphisms is analysed in the last section. For now we wish to stress how we obtain the main ingredients for the proof. The novelty of our approach lies in the remark that, differently from strong shadowing conditions typically considered in dynamics topological works, the (weak) shadowing property necessary for our theorems (and also for the results in [6]) just needs to hold for points in total probability sets, with respect to each system we consider. By *Mañé's Ergodic Closing Lemma* [9], the kind of shadowing by periodic points we need holds residually in the space of  $C^1$ -diffeomorphisms. Recently, We also have generalized Ergodic Closing Lemma to the context of the space  $NEnd^1(M)$  of nonsingular endomorphisms(see [5]). Assuming that residually in an open set  $\mathcal{U} \subset NEnd^1(M)$  the periodic sets are NUH (respectively, NUE), the same arguments in [6] apply in order to obtain residual hyperbolicity (respectively, residual expansion) in  $\mathcal{U}$ .

In the next section, we detail the context and give exact statements of our theorems.

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## 2. SETTING AND STATEMENT OF MAIN RESULTS

In the sequel,  $M$  will always denote a finite dimensional compact Riemannian manifold and  $NEnd^1(M)$  will denote the set of  $C^1$  *nonsingular endomorphisms* defined in  $M$ . By  $g : M \rightarrow M$  to be a *nonsingular endomorphism* we mean that the derivative  $Dg(p)$  of  $g$  in each point  $p \in M$  is a linear isomorphism. We endow  $NEnd^1(M)$  with the  $C^1$  topology; therefore  $NEnd^1(M)$  is an open subset of the complete space  $C^1(M)$  whose elements are  $C^1$ -endomorphisms defined in  $M$ . We also denote by  $Diff^1(M)$  the set whose elements are  $C^1$  diffeomorphisms on  $M$ .

For the statement of our results, we recall here the notion of hyperbolic set:

**Definition 1.** Let  $\Lambda$  be an invariant set for a  $C^1$  diffeomorphism  $f$  of a manifold  $M$ . We say that  $\Lambda$  is a hyperbolic set if there is a continuous splitting  $T_\Lambda M = E^s \oplus E^u$  which is  $Tf$ -invariant ( $Tf(E^s) = E^s, Tf(E^u) = E^u$ ) and for which there are constants  $c > 0$ ,  $0 < \varsigma < 1$ , such that

$$\|Tf^n|_{E^s}\| < c \cdot \varsigma^n, \quad \|Tf^{-n}|_{E^u}\| < c \cdot \varsigma^n, \forall n \in \mathbb{N}.$$

We also recall here the notion of nonwandering set, and of Axiom A diffeomorphisms:

**Definition 2.** (Nonwandering set.) Let  $X$  be a topological space and  $f : X \rightarrow X$  a continuous map. A point  $x \in X$  is nonwandering if for any neighborhood  $V$  of  $x$ , there exists  $k \in \mathbb{N}$  such that  $f^k(V) \cap V \neq \emptyset$ . We denote the set of all nonwandering points of  $f$  by  $\Omega(f)$ .

**Definition 3.** (Axiom A.) Let  $M$  be a compact manifold. A diffeomorphism  $f : M \rightarrow M$  is *Axiom A* if:

- The nonwandering set  $\Omega(f)$  is a hyperbolic set;
- $\Omega(f) = \overline{Per(f)}$ .

**Definition 4.** (Non uniformly hyperbolic set). Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . We say that an invariant set  $X \subset M$  is a *non uniformly hyperbolic set* or, simply, *NUH*, iff

- (1) There is an  $Df$ -invariant splitting  $T_X M = E^{cs} \oplus E^{cu}$ ;
- (2) There exists  $\lambda < 0$  and an adapted Riemannian metric for which any point  $p \in X$  satisfies

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))|_{E^{cs}(f^j(p))}\| \leq \lambda$$

and

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| [Df(f^j(p))|_{E^{cu}(f^j(p))}]^{-1} \| \leq \lambda$$

**Remark 5.** Sometimes we say that  $f$  is *NUH on  $X$*  meaning that  $X \subset M$  is *NUH* (for  $f$ ).

**Definition 6.** (*Dominated splitting*). Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$  and let  $X \subset M$  be an invariant subset. We say that a splitting  $T_X M = E \oplus \hat{E}$  is a *dominated splitting* iff:

- (1) The splitting is invariant by  $Df$ , which means that  $Df(E(x)) = E(f(x))$  and  $Df(\hat{E}(x)) = \hat{E}(f(x))$ .
- (2) There exist  $0 < \eta < 1$  and some  $l \in \mathbb{N}$  such that for all  $x \in X$

$$\sup_{v \in E, \|v\|=1} \{ \|Df^l(x)v\| \} \cdot \left( \inf_{v \in \hat{E}, \|v\|=1} \{ \|Df^l(x)v\| \} \right)^{-1} \leq \eta.$$

In this paper, we study the consequences for an open set  $\mathcal{U}$  of  $C^1$ -diffeomorphisms, if it has a residual subset  $\mathcal{S}$  in which each map exhibits a NUH periodic set admitting a dominated splitting. In such case, we prove that  $\mathcal{U}$  has a residual subset  $\mathcal{A}$  such that  $f$  is Axiom A, for all  $f \in \mathcal{A}$ .

More precisely we prove:

**Theorem A.** *Let  $\mathcal{U} \subset \text{Diff}^1(M)$  an open subset of diffeomorphisms on a compact manifold  $M$ . Suppose that for any  $f$  in some residual subset  $\mathcal{S}$  of  $\mathcal{U}$ , the set  $\text{Per}(f) \subset M$  of periodic points of  $f$  is non uniformly hyperbolic (NUH), and  $T_{\text{Per}(f)} M = E^{cs} \oplus E^{cu}$  is a dominated splitting. Then, there exists a residual subset of  $\mathcal{U}$  whose elements are Axiom A diffeomorphisms. In particular,  $\mathcal{U}$  is contained in the closure of the Axiom A diffeomorphisms set.*

**Theorem B.** *Let  $\mathcal{U} \subset \text{Diff}^1(M)$  an open subset of diffeomorphisms on a compact manifold  $M$ . Suppose that for any  $g$  in some residual subset  $\mathcal{S}$  of  $\mathcal{U}$ , the set  $\text{Per}(f)$  of periodic points of  $f$  is non uniformly hyperbolic (NUH), and  $T_{\text{Per}(f)} M = E^{cs} \oplus E^{cu}$  is a continuous splitting extending to  $T_{\overline{\text{Per}(f)}} M$ . Then, there exists a residual subset  $\mathcal{A}$  of  $\mathcal{U}$  whose elements are Axiom A diffeomorphisms. In particular,  $\mathcal{U}$  is contained in the closure of the Axiom A diffeomorphisms set.*

In fact, Theorem A is a consequence of Theorem B. Nevertheless, the hypotheses in A are easier to verify.

**Remark 7.** Due to Palis' work (see [11]), Axiom A are not necessarily open if a no-cycles condition is not assumed.

For endomorphisms, the concept of hyperbolicity is slightly different. As the notion of Axiom A endomorphisms is a quite elaborated notion, we treat then as an special case in the last section of this paper. The most standard kind of hyperbolic endomorphism is the expanding one:

**Definition 8.** (Expanding map.) A  $C^1$  map  $g : M \rightarrow M$  on a manifold is *expanding* if there are constants  $C > 0$  and  $\sigma > 1$  such that

$$\|[Dg^n(x)]^{-1}\| < C \cdot \sigma^{-n}, \forall n \in \mathbb{N}.$$

We also recall the non uniformly expanding (*NUE*) definition:

**Definition 9.** (*NUE*.) We say that a map  $g : M \rightarrow M$  is non uniformly expanding (*NUE*) on a set  $X \subset M$  (or, equivalently, that  $X$  is *NUE* for  $g$ ), if there exists  $\lambda < 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(x))]^{-1}\| \leq \lambda < 0 \text{ for all } x \in X.$$

We also obtain the following corresponding result for nonsingular endomorphisms:

**Theorem C.** Let  $\mathcal{S} \subset \mathcal{U}$  a residual subset of an open set  $\mathcal{U}$  contained in  $\text{NEnd}^1(M)$ . Suppose that each  $g \in \mathcal{S}$ , is non uniformly expanding (*NUE*) on its respective set  $\text{Per}(g) \subset M$  of periodic points. Then, there exists an open and dense subset of  $\mathcal{U}$  whose elements are expanding maps.

In the last section of the paper, we also prove the analogous result for Axiom A endomorphisms, whose statement is similar to Th. B.

### 3. PROOF OF THE CRITERIA OF GENERIC HYPERBOLICITY

Let  $f : M \rightarrow M$  be a diffeomorphism in  $\mathcal{U}$ . Along this section, we suppose that the periodic set  $\text{Per}(f)$  is NUH. (see definition 4, on page 3).

**Remark 10.** We note that the set of periodic points  $\text{Per}(f)$  is NUH iff there exists  $\lambda < 0$  such that for each periodic point  $p$  with period  $t(p)$ , then

$$\sum_{j=0}^{t(p)-1} \log(\|[Df|_{E^{cu}}(f^j(p))]^{-1}\|) \leq \lambda \cdot t(p)$$

and

$$\sum_{j=0}^{t(p)-1} \log(\|Df|_{E^{cs}}(f^j(p))\|) \leq \lambda \cdot t(p).$$

Before we prove Th. B, let us introduce some notation and recall some classical results we shall use. The first one is the Closing Lemma, by Pugh [13]. It asserts that there is a residual  $\tilde{\mathcal{R}}$  set of  $\text{Diff}^1(M)$  such that  $\Omega(f) = \overline{\text{Per}(f)}$ ,  $\forall f \in \tilde{\mathcal{R}}$ . The other classical result that we will need is:

**Theorem 11.** (*Ergodic Closing Lemma - Mañé [9].*) *Let  $M$  be a compact manifold. There is a residual  $\mathcal{R} \subset \text{Diff}^1(M)$  such that for each  $f \in \mathcal{R}$ , the set  $\mathcal{M}_1(f)$  of  $f$ -invariant probabilities is the closed convex hull of the ergodic measures supported in the hyperbolic periodic orbits of  $f$ .*

By Oseledets([10]), it is known that if  $\mu$  is an invariant measure for a  $C^1$  map  $g$ , then the number

$$\lambda(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Dg^n(x)v\|,$$

is defined in a set of total probability (that is, a set that has full measure for any  $g$ -invariant probability) and it is called *Lyapunov exponent* at  $x$  in the direction  $v$ . In [1], the author proved that if  $g$  is a nonsingular endomorphism such that for any  $g$ -invariant measure, all Lyapunov exponents are positive then  $g$  is an expanding map. He also obtained analogous results for diffeomorphisms admitting continuous splitting.

We will also use the following technical lemma, which we state a few more general than in its original paper.

**Lemma 12.** [1] *Let  $g : K \rightarrow K$  be a map defined in a compact metric space  $K$ . Let  $\mathcal{M}_1(g)$  be the space of  $g$ -invariant probabilities, and  $\phi$  a continuous function on  $K$ . If  $\int \phi d\mu < \lambda, \forall \mu \in \mathcal{M}_1(g)$ , then there exists  $N \in \mathbb{N}$  such that  $\forall n \geq N$  we have*

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(g^i(x)) < \lambda, \forall x \in K.$$

Given  $g \in \text{NEnd}^1(M)$ , we notice that if the condition in hypothesis of the lemma holds for  $\phi = \log \|Dg^{-1}\|$  then  $g$  is expanding. Analogously, if  $f \in \text{Diff}^1(M)$ , taking  $\phi = \log \|Df|_{E^{cs}}\|, \log \|Df^{-1}|_{E^{cu}}\|$ , assuming that  $E^{cs} \oplus E^{cu}$  is a continuous splitting of  $T_\Lambda M$ ,  $\Lambda =$  compact invariant set), then  $f$  is hyperbolic in  $\Lambda$ . This fact is an immediate consequence of the following simple proposition:

**Proposition 13.** *Let  $g : K \rightarrow K$  be a continuous map defined in a compact metric space  $K$ . Let  $\phi_n : K \rightarrow \mathbb{R}$  be a  $g$ -subadditive sequence of continuous functions (that is,  $\phi_{n_1+n_2}(x) \leq \phi_{n_1}(x) + \phi_{n_2} \circ g^{n_1}(x)$ ,  $\forall n_1, n_2 \in \mathbb{N}$ ). Suppose that  $\int_M \phi_1 d\mu < \lambda$ ,  $\forall \mu \in \mathcal{M}_1(g)$ . Then there exists  $N$  such that for all  $n \geq N$ ,*

$$\phi_n(x) \leq n \cdot \lambda, \forall x \in K.$$

**Proof:** By lemma 12 applied to  $\phi = \phi_1$ , there is  $N \in \mathbb{N}$  such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi_1(g^i(x)) < \lambda, \forall x \in K.$$

The subadditivity of  $\phi_n$  then implies that

$$\phi_n \leq \sum_{i=0}^{n-1} \phi_1(g^i(x)) < n \cdot \lambda, \forall n \geq N, \forall x \in K.$$

□

Now we can proceed in the proof of our main results:

**Proof: (Theorem B)** Let  $f : M \rightarrow M$  be a diffeomorphism belonging to the residual set  $\mathcal{A}$  given by the intersection of  $\mathcal{S}$ ,  $\tilde{\mathcal{R}}$  and  $\mathcal{R}$ , the last ones given respectively in Pugh's and Mañé's Closing Lemmas. Therefore, in particular, we have  $\Omega(f) = \overline{Per}(f)$  and the space of invariant probabilities for  $f$  is the closed convex hull of ergodic probabilities supported in periodic orbits. Put  $\phi_s := \log(\|Df|_{E^{cs}}\|)$ ,  $\phi_u := \log(\|[Df|_{E^{cu}}]^{-1}\|)$ , and take  $K = \Omega(f)$  and  $\phi_1 = \phi_s$ , respectively,  $\phi_1 = \phi_u$ , in Proposition 13 (therefore  $\phi_n = \log(\|Df^n|_{E^{cs}}\|)$ , respectively,  $\phi_n = \log(\|[Df^n|_{E^{cu}}]^{-1}\|)$ ,  $\forall n \in \mathbb{N}$ ). Such functions are continuous, since the subbundles  $E^{cs}$ ,  $E^{cu}$  are continuous. Since  $f \in \mathcal{S}$ , there exists  $\lambda < 0$  such that, for any periodic point  $p$  with period  $t(p)$  we have (see remark 10):

$$\int_{\Omega(f)} \phi_s d\left(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\right) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_s(f^j(p)) \leq \lambda$$

and

$$\int_{\Omega(f)} \phi_u d\left(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\right) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_u(f^j(p)) \leq \lambda,$$

where  $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}$  is the ergodic measure supported in the periodic orbit of  $p$ . Since the thesis of Ergodic Closing Lemma holds for  $f$ , all  $f$ -invariant probability  $\mu$  is the limit of a convex combination of such

measures supported in periodic orbits and so we conclude that

$$\int_{\Omega(f)} \phi_s d\mu \leq \lambda$$

and

$$\int_{\Omega(f)} \phi_u d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(f).$$

(Note that  $\mathcal{M}_1(f|_{\Omega(f)}) \simeq \mathcal{M}_1(f)$ .)

We are under the hypothesis of Proposition 13 (just exchange  $\lambda$  by  $0 > \lambda' > \lambda$  to grant strict inequality in the statement of lemma 12). Hence, we conclude that  $\overline{Per(f)} = \Omega(f)$  is a hyperbolic set for  $f$ .  $\square$

**Remark 14.** Suppose that we are under the same hypotheses of our Th. B (or Th. A). We note that if it occurs that for  $f \in \mathcal{A}$ ,  $\Omega(f)$  has no cycles (see [11]), then we conclude that  $\Omega(\hat{f})$  is hyperbolic for all  $\hat{f}$  belonging in an open and dense subset  $\tilde{\mathcal{A}} \subset \mathcal{U}$ ,  $\tilde{\mathcal{A}} \supset \mathcal{A}$ .

Note that Th. A has the same thesis of Theorem B, but instead of the existence of an invariant splitting over the periodic set continuously extending to its closure, in Th. A we assume the existence of a dominated splitting over the periodic set. So, **Theorem A is a consequence of Th. B and the simple**

**Lemma 15.** [6] *Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . Let  $X \subset M$  be some  $f$ -invariant set. Suppose there exists some invariant dominated splitting  $T_X M = E \oplus \hat{E}$ . Then, such splitting is continuous in  $T_X M$ , and unique once we fix the dimensions of  $E, \hat{E}$ . Moreover, it extends uniquely and continuously to a splitting of  $T_{\bar{X}} M$ .*

**Remark 16.** Just as in the case of remark 10,  $Per(g)$  is NUE iff there exists  $\lambda < 0$  such that for each periodic point  $p$  with period  $t(p)$ , then  $\sum_{j=0}^{t(p)-1} \log(\|Dg(g^j(p))\|^{-1}) \leq \lambda \cdot t(p)$ .

As the reader may guess, the proof of Theorem C follows in the same manner as Ths. A and B (in fact, the proof is even simpler), through the version stated in the introduction (Th. A in [5]) of Ergodic Closing Lemma for Nonsingular Endomorphisms.

**Proof: (Th. C)** Let  $g : M \rightarrow M$ ,  $g \in \mathcal{S} \subset NEnd^1(M)$  belonging in the residual set  $\mathcal{R}$  given by the Residual version of Ergodic Closing Lemma for nonsingular endomorphisms we proved in [5] (Th. A in [5]). Therefore, in particular, we have that the space of invariant probabilities for  $g$  is the closed convex hull of ergodic probabilities supported



in periodic orbits. Put  $\phi := \log(\|[Dg]^{-1}\|)$ . Since  $g \in \mathcal{S}$ , there exists  $\lambda < 0$  such that, for any periodic point  $p$  with period  $t(p)$  we have (see remark 16):

$$\int_M \phi d\left(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}\right) = \sum_{j=0}^{t(p)-1} \phi(g^j(p)) \leq \lambda,$$

where  $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}$  is the ergodic measure supported in the periodic orbit of  $p$ . As the thesis of Ergodic Closing Lemma holds for  $g$ , all  $g$ -invariant probability  $\mu$  is the limit of a convex combination of such measures supported in periodic orbits and so we conclude that

$$\int_M \phi d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(g).$$

This is again the hypothesis of the fundamental proposition 13, which implies (just exchange  $\lambda$  by  $0 > \lambda' > \lambda$  to grant strict inequality in the statement of the proposition 13) that  $g$  is an expanding map.  $\square$

#### 4. AXIOM A ENDOMORPHISMS

In this section, we present the corresponding statement for Axiom A endomorphism of Th. A that we have proved earlier in the diffeomorphism context.

As we asserted in section 2, the concept of Axiom A endomorphisms is not as simple as the concept of Axiom A diffeomorphisms. This is because a set can be a positively invariant set for an endomorphisms without to be also a negatively invariant set. Moreover, if the dimension of unstable space  $E^u(x)$  of a point  $x$  is not the same of the ambient manifold (which is the expanding map case), such space is not uniquely defined, in general: for each choice of negative branch in the pre-orbit of  $x$  we may obtain a different unstable space for  $x$ .

Let  $X \subset M$  and let  $E$  be a subbundle of the tangent bundle  $TM$  restricted to  $X$ . Given a point  $p \in X$ , the cone  $C_a(p)$  of width  $0 < a = a(p) < 1$  around  $E(p)$  by

$$C_a(p) := \{v \in T_p M, \min_{w \in E(p)} \{\angle(v, w) \leq a\}\}$$

We define the cone field  $C_a$  of width  $a : X \rightarrow (0, 1)$  around  $E$  as the map  $X \ni p \mapsto C_a(p)$ . The cone field is continuous if both  $a$  and  $p \mapsto E(p)$  are continuous.

We say that a cone field  $C_a$  around  $E$  and a subbundle  $\hat{E}$  are complementary, if  $E(p) \oplus \hat{E}(p) = T_p M$ , for all  $p$  in the intersection of the

domains of  $E$  and  $\hat{E}$ . We also say that two cone fields  $C_a$  around  $E$ ,  $\hat{C}_a$  around  $\hat{E}$  are complementary if  $E(p) \oplus \hat{E}(p) = T_p M$ , for  $p$  in the intersection of their domains.

**Definition 17.** (Hyperbolic set for nonsingular endomorphisms.) Let  $\Lambda$  be a positively invariant compact set for a  $C^1$  endomorphism  $g$  of a manifold  $M$ . Denote by  $X := \cup_{n=0}^{+\infty} g^{-n}(\Lambda)$ . We say that  $\Lambda$  is a hyperbolic set if there are complementary invariant subbundle  $E^s$  of  $T_\Lambda M$  and a positively invariant cone field  $C_a^u$  defined on  $T_X M$ , such that:

- $E^s$  is  $Tg$ -invariant, that is,  $Dg(E^s(y)) \subset E^s(g(y))$ ,  $y \in \Lambda$ ;
- $C_a^u$  is a  $Tg$ -invariant, that is  $Dg(x) \cdot C_a^u(x) \subset C_a^u(g(x))$ ,  $\forall x \in X$ ;
- The angle between  $E^s(y)$  and  $C_a^u(y)$  is greater than a positive constant,  $\forall y \in \Lambda$ .
- There are constants  $c > 0$ ,  $0 < \varsigma < 1$ , such that

$$\|Dg^n(y)|_{E^s}\| < c \cdot \varsigma^n, \quad \sup_{x \in X, v \in C_a^u(x), \|v\|=1} \{ \| [Dg^n(x) \cdot v]^{-1} \| \} < c \cdot \varsigma^n, \forall n \in \mathbb{N}.$$

**Definition 18.** (Non uniformly hyperbolic set for endomorphisms) Let  $f : M \rightarrow M$  be a nonsingular endomorphism on a compact manifold  $M$ . We say that a positively invariant set  $\Lambda \subset M$  is a *non uniformly hyperbolic set* or, simply, *NUH*, iff

- (1) There are  $Df$ -invariant, complementary subbundle  $E^{cs}$  whose domain is  $\Lambda$  and a cone field  $C^{cu}$  whose domain is  $X := \cup_{n=0}^{+\infty} f^{-n}(\Lambda)$ ;
- (2) There exists  $\lambda < 0$  and an adapted Riemannian metric for which any point  $p \in \Lambda$  satisfies

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))|_{E^{cs}(f^j(p))}\| \leq \lambda$$

and

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| [Df(f^j(p))|_{C^{cu}(f^j(p))}]^{-1} \| \leq \lambda$$

**Theorem 19.** Let  $\mathcal{U} \subset \text{NEnd}^1(M)$  be an open subset of nonsingular endomorphisms on a compact manifold  $M$ . Suppose that for any  $g$  in some residual subset  $\mathcal{S}$  of  $\mathcal{U}$ , the set  $\text{Per}(g)$  of periodic points of  $f$  is non uniformly hyperbolic (NUH). Assume also that both the subbundle  $E^{cs}$  and the cone field  $C^{cu}$  (in definition of NUH) respectively extend continuously to  $\overline{\text{Per}(f)}$  and to  $\overline{\cup_{n=0}^{+\infty} g^{-n}(\text{Per}(g))}$ . Then, there exists a residual subset  $\mathcal{A}$  of  $\mathcal{U}$  whose elements are Axiom A endomorphisms. In particular,  $\mathcal{U}$  is contained in the closure of the Axiom A endomorphisms set.

**Proof:**

Let  $f : M \rightarrow M$  be a nonsingular endomorphism belonging to the residual set  $\mathcal{A}$  given by the intersection of  $\mathcal{S}$ ,  $\tilde{\mathcal{R}}$  and  $\mathcal{R}$ , the last ones given by the Closing Lemmas for Endomorphisms. Therefore, in particular, we have  $\Omega(f) = \overline{Per(f)}$  and the space of invariant measures for  $g$  is the closed convex hull of ergodic measures supported in periodic orbits. Put  $\phi_s := \log(\|Df|_{E^{cs}}\|)$ ,  $\phi_u := \log(\|[Df|_{C^{cu}}]^{-1}\|)$ . Such functions are continuous, since the subbundle  $E^{cs}$  and the cone field  $C^{cu}$  are continuous. Since  $f \in \mathcal{S}$ , there exists  $\lambda < 0$  such that, for any periodic point  $p$  with period  $t(p)$  we have (just as in remark 10):

$$\int_M \phi_s d\left(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\right) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_s(f^j(p)) \leq \lambda$$

and

$$\int_M \phi_u d\left(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\right) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_u(f^j(p)) \leq \lambda,$$

where  $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}$  is the ergodic measure supported in the periodic orbit of  $p$ . Since the thesis of Ergodic Closing Lemma holds for  $g$ , all  $g$ -invariant probability  $\mu$  is the limit of a convex combination of such measures supported in periodic orbits and so we conclude that

$$\int_M \phi_s d\mu \leq \lambda$$

and

$$\int_M \phi_u d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(f).$$

We are under the hypothesis of proposition 13 (just exchange  $\lambda$  by  $0 > \lambda' > \lambda$  to grant strict inequality in the statement of lemma 12). Hence, we conclude that  $\overline{Per(f)} = \Omega(f)$  is a hyperbolic set for  $f$ .  $\square$

## REFERENCES

- [1] Y. Cao, *Non-zero Lyapunov exponents and uniform hyperbolicity*, Nonlinearity **16** (2003), 1473-1479.
- [2] C. Bonatti, L. Díaz and E.R. Pujals. *A  $C^1$ -generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources*, Annals of Math., 158(2), 355-418, (2003).
- [3] A. Castro, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors*, Israel J. Math. **130** (2002), 29-75.
- [4] A. Castro, *Fast mixing for attractors with mostly contracting central direction*, Ergodic Th. Dynam. & Syst., **24** (2004), 17-44.

- [5] A. Castro, *The Ergodic Closing Lemma for Nonsingular Endomorphisms*, ArXiv, 0906.2031 (2009).
- [6] A. Castro, K. Oliveira, V. Pinheiro. *Shadowing by nonuniformly hyperbolic periodic points and uniform hyperbolicity*, Nonlinearity **20** (2007), 75-85.
- [7] L. Díaz and F. Abdenur. *Pseudo-orbit shadowing in the  $C^1$ -topology*, submitted to AIMS' Journals.
- [8] K. Lee, K. Moriyasu, K. Sakai,  *$C^1$ -stable shadowing diffeomorphisms*, Discrete and Continuous Dynamical Systems, vol. 22, 3, (2008), pp. 683-697.
- [9] R. Mañé, *An Ergodic Closing Lemma*, Annals of Math. **116**, (1982), 503-541.
- [10] V. I. Oseledets, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19, (1968), 197-231.
- [11] J. Palis, *A note on  $\Omega$ -stability*, Global Analysis, Proc. Symp. Pure Math., A.M.S., XIV (1970), 221-222.
- [12] V. Pinheiro, *Sinai Ruelle Bowen measures for weakly expanding maps*, Nonlinearity (Bristol), United Kingdom, v. 19 (2006), p. 1185-1200.
- [13] C. Pugh, *The closing lemma*, Amer. J. Math. **89** (1967), 956-1009.
- [14] J. Schmeling, S. Troubetzkoy, *Dimension and invertibility of hyperbolic endomorphisms with singularities*, Ergod. Th. Dynam. & Syst., **18** (1998), 1257-1282.
- [15] L. Wen, *The  $C^1$ -Closing Lemma for nonsingular endomorphisms*, Ergod. Th & Dynam. Sys. **11**, (1991) 393-412.

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